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Fuzzy Approach for Bi-Level Programming Problems

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ABSTRACT. Bi-level programming problem (BLPP) can make the final decision result is a mutually coordinated scheme among all layers, which can solve complex practical problems well. This paper summarizes and describes the methods to solve the bi-level programming problem, and proposes a fuzzy method to solve the BLPP by using the concepts of membership function and multi-objective optimization to solve the bilevel programming problem, and gives a numerical example to solve it. It also points out that developing a method to solve fuzzy problems and new concepts to solve multi-level programming problems are practical directions for future research.

Keywords: bi-level programming, kth-best algorithm, fuzzy approach

1. Introduction

To reflect the tradeoff between two decision-making levels with different concerns, the bi-level programing (BP) approach with a two-level structure was developed (Camacho, 1997). A leader-follower decision-making strategy was incurporated into the optimization process of BP to generate satisfied decision-making plan for both the upper-level and the lower-level decision makers. When dealing with a system problem, different decision-makers (DMs) generally have different perspectives and starting points, which leads to conflicts in the design of ideal solutions for certain variables. Although the BP algorithm can effectively balance the needs of decision-makers at different levels, it has limitations in representing uncertain parameters in the planning system. Therefore, different types of fuzzy numbers, interval numbers and BP linear programming can be combined to address the uncertainty in the system, and a suitable algorithm can be used to convert the fuzzy number and interval number into clear values and obtain an effective optimization plan (Ma et al., 2017).

Many approaches have been proposed to solve MLPP. Most of these methods are based on conceptual approaches of vertex enumeration and transformation. The former is to find the control variable with a higher level of compromise point using the adjusted simplex algorithm. It is inefficient, especially for large problems. Although there is a shortcut, the generality will be lost. The latter involves modifying the programming problem condition or penalty function of the upper-level constraint constructed by Kuhn-Tucker (K-T) at the lower level. Because nonlinearities or Lagrangian terms appear in the constraints, the auxiliary problem becomes complex and sometimes unmanageable.

Aiming at the problem of low computational efficiency of the existing methods, a fuzzy method is proposed to solve the above problems by using the concepts of membership function and multi-objective optimization. The solution search relies on changes in membership functions that represent the satisfaction of possible solutions for decisions at the upper and lower levels, rather than vertex enumeration, and does not generate higher-order constraints. Unlike vertex enumeration, we do not assume a priori that the optimal solution exists at the corner points. On the other hand, we argue that a satisfactory notion is more acceptable than optimality, since it is difficult to define a reliable optimality in a multi-person decision process, and restricting the potential solutions of corner points is by definition problematic. Potentially satisfactory solutions are those in the non-dominated region. Therefore, the proposed method is very efficient and does not increase the complexity of the original problem. Then, Bard's grid search algorithm (Bard, 1983), Wen and Hsu's bi-criteria algorithm (Wen and Hsu, 1989), and the two-stage method (Wen and Hsu, 1992) are used to solve the auxiliary MLPP.

Therefore, how to solve the fuzzy method of bi-level programming problem (BLPP) is the main content of this paper. We first provide a historical review of traditional approaches. In Section 2, we further discuss the Kth-best approach and propose our fuzzy concept and approach for solving BLPP. To demonstrate our concept, numerical examples are solved and their solutions are compared with classical solutions. Finally, Section 4 draws some concluding remarks and proposes future studies.

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2. Methodology

2.1 The Traditional Method

Bi-level programming is a mathematical optimization model with a two-level hierarchical structure. The upper and lower level problems have their own objective functions and constraints. The objective function and constraints of the upper-level problem are not only related to the upper-level decision variables, but also depend on the optimal solution of the lower-level problem, which in turn is affected by the upper-level decision variables. It can simultaneously consider the interests of both the global and the individual, ensure that the global is first and the local is second, and the final decision result is often a mutually coordinated scheme between all layers, which can well solve complex practical problems (Lv, 2009).

To meet these strategies and reactions, Bialas and Karwan (Bialas and Karwan, 2156) propose the following Kth-best algorithm: the solution search starts at the individual optimum of the upper-level DM, and the optimal solution is reached if it matches the lower-level DM's optimality; otherwise, search for the neighboring corner (extreme) points of the previous point until the upper-level DM's proposed decision matches the lower-level DM's optimality. Through this algorithm, we can see how the upper-level DM decreases his or her objective value in order to make a compromise for the lower-level DM's optimality. In fact, an implicit compromise process has been carried out through the solution search. Let us consider an extreme situation where the independent solutions for two DMs are located at two neighboring vertices. The Kth-best algorithm will force its solution to be either of them, depending on who goes first; i.e., the DM who moves first absolutely dominates the solu-tion. Since the conflict has not been solved yet, the Kth-best's solution seems less meaningful. A compromise solution between these two extreme points should be more practical. Indeed, this phenomenon happens in all problems. The fundamental of the Kth-best algorithm is that the optimum should exist among corner points and that the corner point search leads to complicated enumeration. As mentioned above (Basar, 1982; Jenkins and Passino, 1999), it is difficult to define a solid optimality for multi-person, decision-making problems. Compromise or coordination are usually needed in order to reach a solution, even in a non-cooperative phenomenon.

For large-scale problems, the Kth-Best algorithm is quite slow to reach the desired solution because the simplex algorithm has to search a large number of vertices. Second, DMS do not know the relationship between levels and the possible impact of individual operations on each other, that is, there is a lack of clear information, especially about the achievement of goals for each DM. In the third point, the program shows that the rate of return always occurs on the upper DM, while the lower DM makes some profit when exploiting the losses of the former, that is, declining sequentially from the upper best if the lower level is not satisfied. It clearly violates the law of nature, boss first, in the hierarchy. Finally, even in a decentralized organization, a non-dominant solution may make more sense than a classical one. In this case, other non-angular, non-dominated solutions may be good enough while avoiding the computational difficulties caused by enumeration. In many other cases, the Kth-Best solution dominates and is therefore not very attractive to either DM, especially in practice.

2.2 Fuzzy Approach for BLPP

Instead of searching through vertices as the Kth-best algorithm, we here propose a supervised search procedure (supervised by top-level DM) which will generate a (non-dominated) satisfactory solution for a multi-level programming problem. In this solution search, the upper-level DM specifies preferred values of his or her control variables and goals with some leeway. This information is modeled by membership functions of fuzzy set theory and passed to the lower-level DM as his or her additional constraints or boss's requirements. The lower-level DM should not only optimize his or her objective but also try to satisfy the upper-level DM's goal and preference as much as possible. He or she realizes that without seriously considering the boss's goal and preference, the proposed solution will very possibly be rejected and the solution search will be a lengthy one. The lower-level DM then presents his or her solution to the upper-level DM. If the upper-level DM agrees to the proposed solution, a solution is reached and it is called a satisfactory solution here. If he or she rejects this proposal, the upper-level DM will need to re-evaluate and change former goals and decisions as well as their corresponding leeway or tolerances until a satisfactory solution is reached. This strategy does not violate the non-cooperate nature-both level DMs first seek their optimal solutions in isolation. However, it does need some kind of coordination with the classical approaches.

Mathematically, the upper-level DM first solves the following problem:

$$\max f_1(x_1, x_2) = c_{11}x_1 + c_{12}x_2 \tag{1}$$

$$s.t.\quad (x_1, x_2) \in F_2 = \left\{(x_1, x_2) \middle| A_1x_1 + A_2x_2 \le b_1x_1 \ and\ x_2 \ge 0\right\},$$

whose solution is assumed to be (x_1^U, x_2^U, f_1^U), and the lower-level DM independently solves:

$$\max f_2(x_1, x_2) = c_{21}x_1 + c_{22}x_2 \tag{2}$$

$$s.t.\quad (x_1, x_2) \in F_2 = \left\{(x_1, x_2) \middle| A_1x_1 + A_2x_2 \le b_1x_1 \ and\ x_2 \ge 0\right\},$$

whose solution is assumed to be (x_1^L, x_2^L, f_2^L). The above solutions are then disclosed to both DMs. If (x_1^U, x_2^U) = (x_1^L, x_2^L), an optimal solution is reached. However, two solutions are usually different because of conflicts of nature between two objectives. The upper-level DM understands that using the optimal decision x_1^U as a control factor for the lower-level DM is obviously not practical. It is more reasonable to have some leeway or tolerances that give the lower-level DM a wider feasible domain to search for his or her optimal solution, and that will significantly reduce searching time or iterations. The range of the decision on x_1 should be "around x_1^U with its maximum tolerances P_1". The most preferred decision is x_1^U; the worst acceptable decision is at $x_1^U - P_1$ and $x_1^U + P_1$, and that satisfaction or preference is linearly increasing within the interval of $[x_1^U - P_1, x_1^U]$, and linearly decreasing within $[x_1^U + P_1, x_1^U]$, and other decisions are not acceptable. This information can then be formulated as the following membership functions of fuzzy set theory (Zhang et al., 2019):

$$\mu_{x_1}(x_1) = \begin{cases} \dfrac{\left[x_1 - (x_1^U - P_1)\right]}{P_1}, & if\ x_1^U - P_1 \le x_1 \le x_1^U \\ \dfrac{\left[(x_1^U + P_1) - x_1\right]}{P_1}, & if\ x_1^U - x_1 \le x_1^U + P_1 \\ 0, & otherwise \end{cases} \quad (3)$$

Which is also depicted in Figure 2.

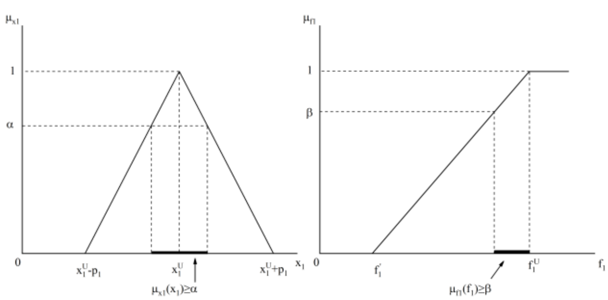

Figure 1. The membership functions for x_1 and f_1.

Meanwhile, it is very important that the upper-level DM should specify his goal with his or her tolerance to the lower-level DM in order to direct or supervise him or her to search for solutions in the right direction. The upper-level DM's goal may reasonably consider that all $f_1 \ge f_1^U$ are absolutely acceptable and all $f_1 < f_1'$ $[=f_1(x_1^L, x_2^L)]$ are absolutely unacceptable, and that the preference within $[f_1', f_1^U]$ is linearly increasing. The fact that the lower-level DM obtained the optimum at (x_1^L, x_2^L), which in turn provides the upper-level DM the objective value of f_1', makes any $f_1 < f_1'$ unattractive in practice. The following membership function can then be reasonably assumed:

$$\mu_{f_1}[f_1(x_1)] = \begin{cases} 1, \quad if\ f_1(x) > f_1^U \\ [f_1(x) - f_1'] / [f_1^U - f_1'], \quad if\ f_1' \le f_1(x) \le f_1^U \\ 0, \quad if\ f_1(x) < f_1 \end{cases} \quad (4)$$

which is also illustrated in Figure 2. The lower-level DM now optimizes his objective under the new constraints of "x_1 is about x_1^U" and "f_1 is somehow near to or greater than f_1^U" which are modeled by the membership functions (3) and (4). The lower-level DM then obtains the following problem (Lai and Hwang, 1993):

$$\max_{x_2} f_2(x_1, x_2) = c_{21}x_1 + c_{22}x_2 \quad (5)$$

$$s.t. \quad A_1x_1 + A_2x_2 \le b,$$

$$x_1 = x_1^U,$$

$$f_1(x) \ge f_1^U,$$

x_1 and $x_2 \ge 0$

Or

$$\max_{x_2} f_2(x_1, x_2) = c_{21}x_1 + c_{22}x_2 \quad (6)$$

$$s.t. \quad A_1x_1 + A_2x_2 \le b,$$

$$\mu_{x_1}(x_1) \ge \alpha I,$$

$$\mu_{f_1}[f_1(x)] \ge \beta,$$

x_1 and $x_2 \ge 0$,

$\alpha \in [0, 1]$ and $\beta \in [0, 1]$

where α (a row vector) and β are minimum acceptable degrees of satisfaction or preference for the decision x_1 and objective, respectively, and I is a column vector with all elements equal to 1s and the same dimension as $\mu_{x1}(x_1)$ or x_1. The feasible ranges constrained by $\mu_{x1}(x_1) \ge \alpha$ and $\mu_{f1}[f_1(x)] \ge \beta$ are depicted in Figure 2. Obviously, the lower-level DM can analyze various solutions corresponding to the upper-level DM's satisfactory levels α and β.

For each possible solution available to the upper-level DM, the lower-level DM may be willing to build a membership function for his or her objective so that he or she can rate the satisfaction of each potential solution. Here, assume that the lower-level DM has the following membership function for his goal:

$$\mu_{f_2}[f_2(x)] = \begin{cases} 1, & if\ f_2(x) > f_2^U \\ [f_2(x) - f_2'] / [f_2^U = f_2'], & if\ f_2' \le f_2(x) \le f_2^U \\ 0, & if\ f_2(x) < f_2' \end{cases} \quad (7)$$

where $f_2' = f_2(x_1^T)$. Obviously, the above membership function μ is a one-to-one mapping within a compact interval of f_2^L and f_2'. Because f_2^L is the best solution of (6), $f_2(x_1) > f_2^L$ is impossible while the upper-level DM gives more constraints to the lower-level DM. The lower-level DM will not accept any $f_2(x) > f_2'$ for the same reason as the upper-level DM, discussed above. Therefore, the lower-level DM has $\mu_{f2}[f_2(x)] = [f_2(x) -$

f_2'] / [f_2^U – f_2'] and the following auxiliary model:

$$\max \delta = \mu_{f_2}[f_2(x)] \quad (8)$$

$$s.t. \quad A_1x_1 + A_2x_2 \le b,$$

$$\mu_{x_1}(x_1) \ge \alpha I,$$

$$\mu_{f_1}[f_1(x)] \ge \beta,$$

x_1 and $x_2 \ge 0$,

$\alpha \in [0, 1]$ and $\beta \in [0, 1]$

Or

$$\max \delta \quad (9)$$

$$s.t. \quad A_1x_1 + A_2x_2 \le b,$$

$$\mu_{x_1}(x_1) \ge \alpha I,$$

$$\mu_{f_1}[f_1(x)] \ge \beta,$$

$$\mu_{f_2}[f_2(x)] \ge \delta,$$

x_1 and $x_2 \ge 0$,

$\alpha \in [0, 1]$ and $\beta \in [0, 1]$

where δ is the satisfactory degree of the lower-level DM who searches for a solution with a higher δ value under the consideration of α and β values. To resolve conflict between both DMs and to avoid the upper-level DM's rejection, the lower-level DM should try to maximize α, β, and δ simultaneously, that is:

$$\max\{\delta, \alpha, \beta\} \quad (10)$$

$$s.t. \quad A_1x_1 + A_2x_2 \le b,$$

$$\mu_{x_1}(x_1) \ge \alpha I,$$

$$\mu_{f_1}[f_1(x)] \ge \beta,$$

x_1 and $x_2 \ge 0$,

$\alpha \in [0, 1]$ and $\beta \in [0, 1]$

If the rain operator is used to aggregate the satisfactory levels or $\lambda = \min\{\alpha, \beta, \delta\}$, the above problem will become:

$$\max \lambda \quad (11)$$

$$s.t. \quad A_1x_1 + A_2x_2 \le b,$$

$$\mu_{x_1}(x_1) \ge \lambda I,$$

$$\mu_{f_1}[f_1(x)] \ge \lambda,$$

$$\mu_{f_2}[f_2(x)] \ge \lambda,$$

x_1 and $x_2 \ge 0$,

$\lambda \in [0, 1]$

Or

$$\max \lambda \quad (12)$$

$$s.t. \quad A_1x_1 + A_2x_2 \le b,$$

$$[(x_1^U + p_1) - x_1] / p_1 \ge \lambda I,$$

$$[x_1 - (x_1^U - p_1)] / p_1 \ge \lambda I,$$

$$\mu_{f_1}[f_1(x)] = [f_1(x) - f_1'] / [f_1^T - f_1'] \ge \lambda,$$

$$\mu_{f_2}[f_2(x)] = [f_2(x) - f_2'] / [f_2^L - f_2'] \ge \lambda,$$

x_1 and $x_2 \ge 0$,

$\lambda \in [0, 1]$

Equation (9) is actually a fuzzy or max-min programming problem by applying Bellman and max-min decision (Ren and Zhang, 2018).

If the upper-level DM is satisfied with the solution of (9), a satisfactory solution is reached. Otherwise, he or she should provide new membership functions for the control variable and objective to the lower-level DM until a satisfactory solution is reached. Combined with set of control decisions and goals with tolerance, this solution becomes a satisfactory solution.

As to membership functions, the linear (and triangular) forms are chosen for computational efficiency. Other membership functions such as piecewise, exponential, hyperbolic, inverse hyperbolic or some specific power functions may be needed for pragmatical reasons. Many of these nonlinear functions can be transferred into equivalent linear forms by variable transformations. Therefore, linear forms are only discussed here without losing generality. Indeed, membership functions are essential while applying fuzzy approaches to solve real-world problems. They are generated basically by heuristic determination, reliability concerns, theoretical demand and human perception. It is not the purpose of this study to discuss various function forms and methods to generate membership functions.

Concise discussion on these topics has been given by Lai and Hwang (Lai and Hwang, 1993).

3. The numerical example

To illustrate this approach, let us consider the following example.

One export-oriented country is concentrating on two important products, 1 and 2, which are manufactured by ABC company on given capabilities. Product 1 yields a profit of $1 per piece and product 2 a profit of $2 per piece. Product 1 can be exported, yielding a revenue of $2 per piece from foreign countries, while product 2 needs the imported raw materials of $1 per piece. There are two level DMs related to this case, Le. government (upper-level) and the manager of the company (lower-level), and each one can handle one decision variable only, x_1 and x_2, respectively. Two objectives are established respectively: (i) effect on the balance of trade $f_1(x)$, i.e., maximum amount of exports; and (ii) profit on the product $f_2(x)$, i.e., maximum profit. The problem can then be formulated as:

$$\max f_1 = 2x_1 - x_2 (\textit{effect on the export trade}) \tag{13}$$

where x_2 solves:

$$\begin{aligned} \text{s.t. } & 3x_1 - 5x_2 \le 15 \quad (\textit{capacity}), \\ & 3x_1 - x_2 \le 21 \quad (\textit{management}), \\ & 3x_1 + x_2 \le 27 \quad (\textit{space}), \\ & 3x_1 + 4x_2 \le 45 \quad (\textit{material}), \\ & x_1 + 3x_2 \le 30 \quad (\textit{labor hours}), \\ & x_1 \textit{ and } x_2 \ge 0 \end{aligned}$$

whose constraint set is denoted by X. The Kth-best solution is $(x_1, x_2) = (8, 3)$ at K = 2. In addition, the optimum for the upper-level objective is $f_1 = 13.5$ at (7.5, 1.5) and for the lower-level objective is $f_2 = 21$ at (3, 9). The decision variable and objective function spaces are shown in Figure 2.

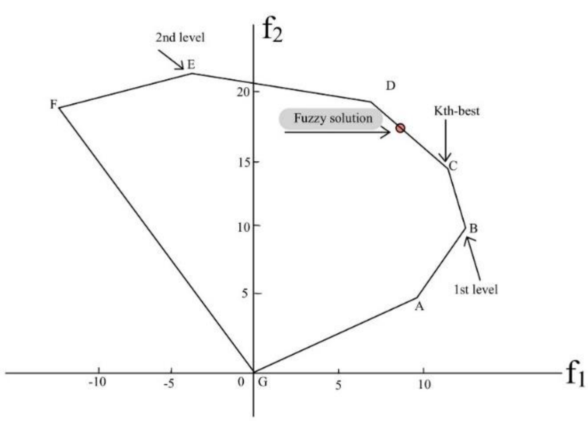

Figure 2. The objective function space for Example.

The proposed approach first finds individual optimal solutions by solving (5) and (6) and obtains $(x_1^U, x_2^U) = (7.5.1.5)$ and $f_1^U = 13.5$ and $x_1^L, x_2^L = (3,9)$ and $f_1^L = 21$. $f_1^T = 13.5$ and let us assume $f_1' = 0$ (only positive is meaningful here) instead of -3 and $f_2' = 10.5$. Assume the upper-level DM's control decision x_1 is around 7.5 with the negative and positive-side tolerances 4.5 and 0.5, respectively. By (3), (4) and (6), membership functions $\mu_{x1}(.)$, $\mu_{f1}(.)$ and $\mu_{f2}(.)$ are built. The lower-level DM then solves the following problem of (9):

$$\max \lambda \tag{14}$$

$$\begin{aligned} s.t.\ & x \in X, \\ & x_1 \ge 4.5\lambda + 3, \\ & x_1 \le 8 - 0.5\lambda, \\ & 2x_1 - x_2 \ge 13.5\lambda, \\ & x_1 + 2x_2 - 10.5 \ge 10.5\lambda, \\ & \lambda \in [0,1] \end{aligned}$$

whose compromise solution is $f^* = (f_1^*, f_2^*) = (9.29, 17.72)$ at $x^* = (7.26, 5.23)$ with the overall satisfaction of both DMs $\lambda = 0.69$. Realized satisfactory levels are $(\mu_{x1}^*, \mu_{f1}^*, \mu_{f2}^*) = (0.95, 0.69, 0.69)$. If the upper-level DM's total satisfactory level $\lambda_1 = \min\{\mu_{x1}^*, \mu_{f1}^*\}$, then our solution provides $\lambda_1 = 0.69$ and λ_2 (of the lower-level DM) = 0.69. On the other hand, the Kth-best solution $f = (13, 14)$ at $x = (8,3)$ has $(\mu_{x1}, \mu_{f1}, \mu_{f2}) = (0.0, 0.96, 0.33)$ and thus $\lambda_1 = 0.0$ and $\lambda_2 = 0.33$. Obviously, our solution is better than that of the Kth-best in terms of satisfactions of both DMs.

4. Conclusion

In this paper, the previous methods for solving bi-level programming problem are summarized and described, a fuzzy method for solving bi-level programming problem is proposed, and a numerical example is given to solve it. The method proposed in this paper to solve BLPP, in the search process does not rely on the enumeration of vertices, but on the change of membership functions, so even large-scale problems can be solved with less computation. For nonlinear programming problems, the proposed method at least does not increase the order of the nonlinearity.

Since the ones mentioned in Section 3 provide satisfactory solutions for member functions and operations, it is important to explore various functions and operators, as well as to allow DMs to change functional forms and operators during the interactions discussed above; Therefore, we can build a complete decision support system for solving MLPP. At the same time, we should also extend our method to solve nonlinear, integer or mixed integer (nonlinear) multi-level programming problems. The input data or parameters are often imprecise or fuzzy. Developing methods to solve fuzzy problems and new concepts to solve multi-level programming problems is a practical and interesting direction for future research.

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